MATH 4460-5460-7460

Advanced Number Theory: ANALYTIC NUMBER THEORY I

Fall 2022 (7-Sep-2022 to 09-Dec-2022)

1 Course

1.1 General Information

Instructor	Habiba Kadiri, PhD (she/her) Associate Professor of Mathematics office SA8348 habiba.kadiri@uleth.ca Webpage - Linked In
Lectures	Tuesday-Thursday: 09:00–10:15 am online: Zoom, Meeting ID: XXX, Passcode: XXX in person: in UH B716 Lectures will be delivered in person, streamed and recorded live via Zoom.
Office Hours	Tuesday-Thursday 9:00am–10:00am, on Zoom (same link as lectures) or by appointment

Course description: Course description for *Advanced Number Theory* (Math 4460-5460-7460) is: Topics in analytic and algebraic number theory, elliptic curves, and modular forms.
Note that this course is cross-listed with the undergraduate program.
Description for this topic *Analytic Number Theory I* is: introduction to the study of prime numbers and of the Riemann zeta function, with applications to explicit results. (see the Abstract for details)

Prerequisite: Elementary Number Theory, Real and Complex Analysis.
Specific Prerequisite for U. of Lethbridge's undergraduate students: Mathematics 3461 (Elementary Number Theory). Some background in analysis (Math 3500, Math 2565, or Math 2570) and complex variables (Math 3560) will be required for this course.

Textbook: I shall make use of the following references (in alphabetical order by author):
Introduction to analytic number Theory, by T. M. Apostol,
Multiplicative number theory (3rd Edition), by H. Davenport,
Introduction to Analytic Number Theory, Lecture Notes by A. J. Hildebrand (University of Illinois),
Multiplicative Number Theory 1. Classical Theory, by H. L. Montgomery and R. C. Vaughan,
Problems in Analytic Number Theory, by M. R. Murty.

1.2 Course objectives and content

1.2.1 Course Abstract

This is a first course in analytic number theory. In this course we will focus on the theory of the Riemann zeta function and of prime numbers. The goal of this course will include proving explicit bounds for the number $\pi(x)$ of primes which are less than a given number x. Building analytical tools to prove the prime number theorem (PNT) will be at the core of this course. We will explore and compare explicit formulas, whether they are using smooth weights or truncated Perron's formula, to relate averages over primes and $\pi(x)$ to sums over the zeros of zeta. Another originality of this course will be to explore each topic explicitly (essentially by computing all the hidden terms implied in the asymptotic estimates). With this respect, students will get an introduction to relevant programming languages and computational software.

This will be closely connected to Analytic Number Theory 2 proposed by Greg Martin (UBC), as the sequences of topics are coordinated between us; the intention is for students at all PIMS institutions to be able to take the second analytic number theory course as a continuation of the first one with maximum benefit. In addition, these two courses will provide excellent training for students who plan to attend the "Inclusive Paths in Explicit Number Theory" CRG summer school in 2023. All these events are part of the PIMS CRG "L-functions in Analytic Number Theory" 2022-2015.

1.2.2 Background

A basic question is whether a given integer is prime or not. A related question is how many prime numbers are there less than a given number x. Calling this number $\pi(x)$, Chebyshev established in 1852 that there exist positive constants c_1 and c_2 satisfying $c_1 < 1 < c_2$ such that $c_1 \frac{x}{\log x} \leq \pi(x) \leq c_2 \frac{x}{\log x}$ for $x \geq 2$. This used elementary sieve methods. Classical analytic number theory spawned from Dirichlet's theorem in 1837 (showing that there are infinitely many prime numbers in any given arithmetic progression), and from Riemann's memoir in 1859 (including some analytic study of the zeta function, the explicit formula relating zeros of zeta to primes, and his famous conjecture on the location of these zeros). In their works they related primes to holomorphic functions called L-functions, denoted $L(s)$. Their work shows that the number $\pi(x)$ of primes less than or equal to x, together with related prime counting functions can be computed in terms of $L(s)$.

In this course we will focus on the theory of the Riemann zeta function and of prime numbers. In addition, throughout the course, each topic will be explored numerically.

1.2.3 Table of Contents

- Arithmetic functions: Basic examples; Additive and multiplicative functions; Moebius function; Euler phi function; von Mangoldt function; divisor and sum-of-divisors functions; Dirichlet convolution;

- Asymptotic estimates: Big oh and small oh notation, asymptotic equivalence; Euler's summation formula; summation by parts; convolution; Dirichlet hyperbola method

- Distribution of primes - elementary methods: Chebyshev type estimates; Mertens type estimates; Elementary consequences of the PNT; PNT and averages of the Moebius function;

- Dirichlet series and Euler products: Dirichlet series of convolution products; Euler product identity; examples; Dirichlet series and summatory functions (Mellin transform representation of Dirichlet series, analytic continuation of the Riemann zeta function); Inversion formulae (Perron formula);

- The Riemann zeta function: basic properties; upper bounds; lower bounds and zero-free region; functional equation; approximate functional equation;

- The Prime Number Theorem: proofs via truncated Perron formula and via Guinnand-Weil explicit formula; consequences of the PNT with error term.

1.2.4 Hybrid format

This course will be delivered in the hybrid format. The current room booked for this course (UH B716) allows to live stream board lectures. The lectures will be streamed and recorded on Zoom, so as to be accessible to students from all time zones or who have a conflict of schedule. In order to give the same experience for both online and in person students, we are requesting some TA support to have someone in the room monitoring zoom questions and to be overall in charge of the tech support for both the instructor and the students. I have experience organizing a hybrid event with Alberta Number Theory Days XIII at BIRS in November 2021 (`http://www.birs.ca/events/2021/2-day-workshops/21w2505`): we coordinated the organizing team between online and in person interactions to make sure all participants kept engaged the same way.
All the course material will be posted on Moodle, including course notes and links to the lectures' recordings on the Zoom cloud,. We are also planning to hold office hours only online.

1.2.5 Guest lecture

In order to enhance the experience of the students in this course, we plan to invite experts to offer guest lectures (at least online):

- Olivier Ramaré (CNRS, Marseilles) on explicit inversion formulae, Perron's formula, and consequence on the error term in the Prime Number Theorem.
- Nathan Ng (Lethbridge) on smooth weights and Mellin transforms.

1.3 Evaluation

Evaluation will be homework based, and will be a combination of assignment problems and projects. The plan is to have 6 assignments and a group project in the following format:

- Assignments will have a core set of questions plus additional ones for the graduate students. Graduate students will return individual work while undergraduate students will have the option to return assignments in group.
- Projects will consist in applying computational tools to make methods covered in class completely explicit. These will be open to collaborative work for, and will be optional for undergraduate students.

Projects:
Here is a preliminary list of projects students could choose from: explicit zero-free region, explicit upper and lower bounds for zeta, explicit approximate functional equation, estimates for the error term in the PNT using an explicit truncated Perron formula, or explicit weights. This list is not exhaustive and students will be invited to also submit their own proposal on which course topic they wish to explore numerically.

1.3.1 Introduction to explicit results

The originality of the course will be to explore the above topics through computational activities and projects to understand the limits of some of the classical methods. For instance, working out the details of the terms hidden in big oh notations will allow to students to grasp how big are the values for parameters described under the generic *sufficiently large*, and thus will give a better appreciation of where the problematic are to prove results true *for all* parameter value. Results were all implied constants are worked out are called *explicit*. Students will be exposed to programming languages and computational software such as Python, Sage, and gp/PARI. They will develop the skills necessary to compute concrete examples, find counterexamples, and learn the capabilities and limitations of current computing technology.

1.3.2 Computational support

We are planning to use Sage math as the software to use for computations. As it is based on the Python language, this will be a relevant skill for students to transfer to the real world. Computational projects will help students build some essential programming skills, including acquiring some basic knowledge in optimization techniques. We are proposing to hire 1-2 (depending of the size of the class) students to provide tech support regarding the installation and use of the software, but also to help students write, debug, or accelerate their programs. For instance, some of our local students in our combined degree computer science-math would be competent for this job. As part of our CRG activities, we will be running an online seminar. We are proposing to invite some experts to introduce CRG seminar participants to Sage math. Students will be invited to participate to these seminar sessions.