MATH 740 - 3: Advanced Topics in Mathematics
Moments of L-functions

Department of Mathematics & Statistics,
University of Northern British Columbia

Fall 2023: September 6, 2023 – December 5, 2023

Essentials

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Office Hours: By appointment

Lectures

Monday and Wednesday 8:30am - 9:50am, online via Zoom

Course Description:

This course will cover advanced topics on moments of L-functions. It is intended to follow Analytic Number Theory I taught by Prof. Habiba Kadiri (University of Lethbridge) in Fall 2022 and Analytic Number Theory II taught by Prof. Greg Martin (UBC) in Winter 2023. All three of these courses are part of the current PIMS CRG "L-functions in Analytic Number Theory".

This course will be taught in Fall 2023, every Monday and Wednesday from 8:30am–9:50am pacific time. It will be delivered entirely online via Zoom to all participants. The lectures will be recorded and the recordings will be posted on the course webpage along with the lecture notes and other supplementary materials. The applicant has experience running online network-wide courses, having successfully taught "Algebraic Number Theory" (Fall 2020) for students at both UNBC and the University of Lethbridge.

In this course, we will explore advanced topics in moments of L-functions including approximate functional equations, zero density estimates, mean value estimates for Dirichlet polynomials, large sieve inequalities, Poisson and Voronoi summation formulae, shifted convolution sums, holomorphic modular forms and associated L-functions, trace formulae, and the spectral theory of automorphic forms. In the first part of the course, we will focus on moment results pertaining to the Riemann zeta function. We will cover all the machinery required to establish an asymptotic formula for the fourth moment of the Riemann zeta function on the critical line following the original work of Ingham while also discussing Heath-Brown's improvements on Ingham's result. Moreover, we will establish upper bounds for higher moments of the Riemann zeta function on the critical line. In the second part of the course, we will introduce automorphic L-functions which generalize the class of degree 1 L-functions to which the Riemann zeta function belongs. We study their basic properties, explicit formulae, approximate functional equations and zero free regions. We will then focus on classical automorphic L-functions associated to the automorphic forms on $GL(2)$ (that are either holomorphic or eigenfunctions of the Laplace operators). We will learn about the necessary tools needed to study moments of families of such L-functions such as Voronoi summation formula, Petersson and Kustensov trace formulae, and estimates for shifted convolution sums. We will establish asymptotic formulae for second moments of families of degree 2 automorphic L-functions, and discuss results pertaining to higher moments as time permits. We will also discuss applications of moment estimates to obtaining subconvexity bounds and establishing non-vanishing results for the underlying L-functions.

As part of the PIMS CRG on L-functions in Analytic Number Theory, the course will include up to eight guest lectures by experts in the field (tentatively including Sieg Baluyot for random matrix theory predictions, Nathan Ng for higher

moments of the Riemann zeta function, Caroline Turnage-Butterbaugh for moments of Dirichlet L-functions, Alexandra Florea for moments of L-functions over function fields, Naomi Tanabe for moments of Rankin–Selberg L-functions, Olga Balkanova for spectral theory of automorphic forms, and Rizwanur Khan for spectral reciprocity).

Prerequisites: A graduate course in analytic number theory that includes Dirichlet series and a complex-analytic proof of the prime number theorem (preferably Analytic Number Theory I taught by Kadiri in Fall 2022

References

The main references for this course will be

- A. Ivić, *The Riemann Zeta-Function: Theory and Applications*
- H. Iwaniec and E. Kowalski, *Analytic Number Theory*

Other resources that we will need include:

- E.C. Titchmarsh, *The Theory of the Riemann Zeta-Function*
- H. Iwaniec, *Topics in Classical Automorphic Forms*
- V. Blomer, E. Fouvry, E. Kowalski, P. Michel D. Milićević, and W. Sawin, *The Second Moment Theory of Families of L-functions*

Grading

The evaluation for this course will consist of 2 write-ups (5–10 pages) of specific topics or results related to moments of L-functions and their applications. These assignments will be completed either individually or in pairs depending on the enrolment in the course. Students outside UNBC will receive credit for this course according to the Western Deans Agreement.

Lecture Notes and Recordings:

The lectures will be recorded and the recordings will be posted on the course webpage along with the lecture notes and other supplementary materials.

Academic Regulations

It is the students' responsibility to familiarize themselves with the regulations concerning academic integrity and ensure that their course work conform to the principles of academic integrity. Please read the academic regulations found at:

http://www.unbc.ca/calendar/undergraduate/general/regulations.html. In particular, read sections 40, 41, 42, 43, 44, and 45.

PIMS code of Conduct

We are committed to offering an inclusive, collaborative and safe space for everyone involved in this course. We ask that all participants are committed to supporting equity, diversity and inclusion; we also ask that all participants have read the PIMS Code of Conduct (https://www.pims.math.ca/files/Code_of_conduct_for_events_and_programs_at_PIMS_1.pdf) and agree to follow it.