Introduction to Cohomology of Arithmetic Groups
Academic year 2024-2025

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The most basic example of an arithmetic group is $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, and understanding the cohomology of this group (and its close relatives) will be the basic theme of this course. The cohomology we are interested in can also be identified with that of the locally symmetric space $\Gamma \backslash X$ where, in this case, $X = SL_n(\mathbb{R})/SO(n)$ is a generalization of the (complex) upper half plane. As such, a diverse set of techniques, stemming from geometry, topology, harmonic analysis, and number theory can be used to analyze the situation. After carefully developing the basics of the subject, we will present some of the major developments in this area (mostly from the 1960s-1970s), and then end with an overview of modern directions. The course will be divided into three parts.

1. (Basic constructions) We will describe some of the basic structure of arithmetic groups and their attached symmetric and locally symmetric spaces. We will focus on the case of SL_n and GL_n to keep the Lie theoretic pre-requisites to a minimum. Specifically, we will hope to cover the following topics:
   - Lie theory basics: root systems, parabolic subgroups
   - Lie algebra cohomology, Invariant polynomials and associated differential forms, Results of Chevalley and Kostant
   - Basic geometry of locally symmetric spaces: 'Horospherical' decompositions, parabolic 'ends' of locally symmetric spaces, reduction theory, and compactifications
2. (Foundational results) The works of E. Calabai and A. Weil in the 1950s-1960s related the cohomology of Γ to various rigidity questions. We begin by describing this and then explaining a remarkable circle of ideas stemming from these early considerations, due to Matsushima, Murakami, Garland, and Raghunathan (among others)
   - Differential geometry in the setting of locally symmetric spaces
   - Deformations and Cohomology of discerte groups
   - Cohomology vanishing results of Matsushima (compact case)
   - Extensions to the non-compact case (after Garland, Raghunathan)
3. (Recent results) The works in part (3) paved the way for a variety of new applications and advances in the theory. In the last part of the course, we will survey recent directions, especially connected to number theory and the theory of automorphic forms. We will focus on three (related) topics:
   - Stable cohomology and relations to Dedekind zeta functions
   - Harder's theory of Eisenstein cohomology rationality of values L-functions
   - Hecke algebra considerations and Venkatesh's 'derived' Hecke algebra actions

Organization, Prerequisites, etc.

The course will meet twice a week and be live streamed over zoom for non UofA participants. Notes (both handwritten and typed) will be distributed to the class. Recordings of the class will also be made available for students.

There grades will be based on three homework assignments, concerning the first two parts of the course, and a final project based on student interest. The homework will accommodate students with diverse backgrounds.

We will aim to make this course accessible to students with a basic background in algebra and analysis (at the level of introductory graduate courses) and basic topology (having seen cohomology before would be useful, but is not absolutely essential). Although no specific knowledge from differential geometry, Lie theory, or number theory are required, additional familiarity or interest in these fields will be useful, especially in the latter parts of the course.

We plan to also host several additional speakers throughout the semester at the UofA who will give live-streamed lectures on related topics of current interest.

References for Part 1

[1] Armand Borel, *Introduction to arithmetic groups*, University Lecture Series, vol. 73, American Mathematical Society, Providence, RI, 2019. Translated from the 1969 French original [MR0244260] by Lam Laurent Pham; Edited and with a preface by Dave Witte Morris. MR3970984

[2] ———, *Introduction to the cohomology of arithmetic groups*, Lie groups and automorphic forms, AMS/IP Stud. Adv. Math., vol. 37, Amer. Math. Soc., Providence, RI, 2006, pp. 51–86, DOI 10.1090/amsip/037/02. MR2272919

[3] Armand Borel and Lizhen Ji, *Compactifications of symmetric and locally symmetric spaces*, Lie theory, Progr. Math., vol. 229, Birkhäuser Boston, Boston, MA, 2005, pp. 69–137, DOI 10.1007/0-8176-4430-X_2. MR2126641

[4] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original. MR1324339

[5] Jay Jorgenson and Serge Lang, *$Pos_n(R)$ and Eisenstein series*, Lecture Notes in Mathematics, vol. 1868, Springer-Verlag, Berlin, 2005. MR2166237

[6] ———, *Spherical inversion on $SL_n(\mathbf{R})$*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2001. MR1834111

References for Part 2

[1] Howard Garland, *A finiteness theorem for K_2 of a number field*, Ann. of Math. (2) **94** (1971), 534–548, DOI 10.2307/1970769. MR0297733

[2] Yozô Matsushima, *On the first Betti number of compact quotient spaces of higher-dimensional symmetric spaces*, Ann. of Math. (2) **75** (1962), 312–330, DOI 10.2307/1970176. MR0158406

[3] Katsumi Nomizu, *Lie groups and differential geometry*, Mathematical Society of Japan, Tokyo, 1956. MR0084166

[4] M. S. Raghunathan, *Cohomology of arithmetic subgroups of algebraic groups. I, II*, Ann. of Math. (2) **86** (1967), 409–424; **87 (1967), 279–304**, issn=0003-486X, review= MR0227313, doi=10.2307/1970585,.

[5] ———, *Discrete subgroups of Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. Band 68, Springer-Verlag, New York-Heidelberg, 1972. MR0507234

[6] André Weil, *On discrete subgroups of Lie groups. II*, Ann. of Math. (2) **75** (1962), 578–602, DOI 10.2307/1970212. MR0137793

References for Part 3

[1] Armand Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 235–272 (1975). MR0387496

[2] G. Harder, *Cohomology of Arithmetic Groups*, available on Author's website.

[3] Akshay Venkatesh, *Cohomology of arithmetic groups and periods of automorphic forms*, Jpn. J. Math. **12** (2017), no. 1, 1–32, DOI 10.1007/s11537-016-1488-2. MR3619577

[4] ———, *Derived Hecke algebra and cohomology of arithmetic groups*, Forum Math. Pi **7** (2019), e7, 119, DOI 10.1017/fmp.2019.6. MR4061961