# Math 8810: High-Dimensional Convexity: Covering, Illumination, and Borsuk's Question

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## Course Description

A graduate-level exploration of extremal properties of convex bodies in high dimensions. This course covers the Hadwiger illumination conjecture and Borsuk's partition question, with a special focus on application of classical geometric covering theorems and probabilistic methods.

## Prerequisites

Consent of the instructors based on a review of the student's academic background. Students are expected to have a rigorous undergraduate foundation in linear algebra and vector geometry, real analysis, and introductory probability. Foundational concepts specific to convex geometry will be reviewed throughout the course to ensure a common baseline.

## Format

This course will be delivered in a hybrid format to accommodate both local and network-wide participants. Lectures will be streamed live via Zoom, with the instructors utilizing on-screen writing tools to ensure clear, real-time mathematical exposition.

Students located at the University of Manitoba will attend and participate in person in the primary lecture room. For other participating PIMS network sites with more than one enrolled student, we will request that a dedicated room equipped with videoconferencing hardware be booked by the host institution to facilitate collaborative, group participation.

All lectures will be recorded. Both the screen recordings and the accompanying written notes will be made available to all enrolled students for asynchronous review and reference.

## Guest Lectures

We plan guest lectures by Beatrice Helen Vritsiou on Illumination conjecture (delivered at University of Alberta and streamed online), and by Danylo Radchenko (virtual).

## Evaluation Scheme

The final grade will be determined by the following components:

- **Test 1 (30%):** In-person, 2 hours long. Students will be asked to formulate definitions and/or theorems, provide a short proof of a result covered in class, and solve several short questions based on exercises provided during lectures.
- **Test 2 (30%):** In-person, 2 hours long. Format is identical to Test 1.
- **Virtual Presentation (40%):**
  - 20% evaluated on presentation design and mathematical content.
  - 20% evaluated on presentation delivery and answers to follow-up questions.

## Course Outline

- **Borsuk's Question.** Borsuk's question on partitioning into sets of smaller diameter. Frankl-Wilson theorem. Kahn-Kalai's counterexample. (1 week) [FW81, KK93, Kal15].
- **Covering of Euclidean Space.** Roger's result on covering of Euclidean space by translates of a convex body. Other cases and proofs: lattice coverings, spherical coverings, discretization technique, fractional coverings, hypergraph coverings. (2 weeks) [Nas18, Rog57, Rog58].
- **Illumination Conjecture.** Hadwiger-Boltiansky illumination conjecture. Rogers-Shephard inequality. Asymptotic bounds via Roger's result. Application of boundedness of isotropic constant. (1 week) [BK18, Gua24, HSTV22, CvHMT24, KL25, Sch14].
- **Bodies of Constant Width.** Bodies of constant width and structural properties. Completion of bounded sets to bodies of constant width preserving diameter. Schramm's result on illumination of bodies of constant width and upper bound on Borsuk's numbers. (2 weeks) [MMO19, Sch88].
- **Volume of Constant Width Bodies.** Bounds on volume of bodies of constant width. Blyaschke-Lebesgue problem and Meissner bodies. Construction of bodies of constant width with small volume. (1 week) [ABN$^+$25, ABPR25].
- **Lower Bounds on Illumination.** Exponential lower bound on illumination number of bodies of constant width. Lower bound on Grunbaum's problem of covering a body by balls of smaller diameter. (1 week) [ABPR25a].
- **Probabilistic Construction of sets that are hard to Cover.** General framework for probabilistic construction of hard to cover sets in terms of measurable graphs. (1 week) [ABPR25c].
- **Universal Covers.** Universal covers for sets of diameter 1. Known constructions. Jung's theorem. Sharp asymptotic lower bound on volume of a universal cover. Applications for Borsuk's question. (2 weeks) [ABPR25c, Las82].
- **Illumination of Cap Bodies (Time Permitting).** Illumination of cap bodies via application of ball packing and covering estimates. (1 week) [ABPR25b].

## Aids and Use of Artificial Intelligence (AI)

No external aids, notes, or electronic devices are permitted during the in-person tests. The supervisors will be requested to arrange the proctoring of the in-person tests. For the virtual presentation, the use of generative AI tools (e.g., Gemini, ChatGPT) is permitted solely as a preliminary research aid. AI may not be used to directly write, derive, or generate any of the mathematical content or text for your presentation. Additionally, the use of AI tools in real-time to assist with answers during the Q&A portion of your presentation is strictly prohibited and will be treated as academic misconduct.

## Academic integrity

The University of Manitoba takes academic integrity seriously. As a member of the International Centre for Academic Integrity, the University defines academic integrity as a commitment to six fundamental values: honesty, trust, fairness, respect, responsibility and courage.

The Department of Mathematics, the Faculty of Science and the University of Manitoba all regard acts of academic dishonesty in quizzes, tests, examinations or assignments as serious offences and may assess a variety of penalties depending on the nature of the offence. Acts of academic dishonesty include but are not limited to bringing unauthorized materials into a test or exam, copying from another student, using answers provided by tutors, plagiarism, and examination personation.

Penalties for violation include being assigned a grade of zero on a test or assignment, being assigned a grade of "F" in a course, compulsory withdrawal from a course or program, suspension from a course/program/faculty or even expulsion from the University. Details can be found in the Student Discipline Bylaw and the Faculty of Science Academic Misconduct Penalties.

## References


[ABPR25a] A. Arman, A. Bondarenko, A. Prymak, and D. Radchenko, *Convex bodies of constant width with exponential illumination number*, Discrete Comput. Geom. **74** (2025).

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[ABN+25] A. Arman, A. Bondarenko, F. Nazarov, A. Prymak, and D. Radchenko, *Small volume bodies of constant width*, Int. Math. Res. Not. IMRN (2025).

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[BK18] K. Bezdek and M. A. Khan, *The geometry of homothetic covering and illumination*, Discrete Geometry and Symmetry, 2018.

[CvHMT24] M. Campos, P. van Hintum, R. Morris, and M. Tiba, *Towards Hadwiger's conjecture via Bourgain Slicing*, Int. Math. Res. Not. IMRN **2024** (2024), no. 10, 8282–8295.

[FW81] P. Frankl and R. M. Wilson, *Intersection theorems with geometric consequences*, Combinatorica **1** (1981), 357–368.

[Gua24] Q. Guan, *A note on Bourgain's slicing problem* (2024). arXiv:2412.09075.

[HSTV22] H. Huang, B. A. Slomka, T. Tkocz, and B.-H. Vritsiou, *Improved bounds for Hadwiger's covering problem via thin-shell estimates*, J. Eur. Math. Soc. (JEMS) **24** (2022), no. 4, 1431–1448.

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[Rog58] ———, *The covering of space by spheres*, Mathematika **5** (1958), 29–37.

[Sch14] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, 2nd ed., Cambridge University Press, 2014.

[Sch88] O. Schramm, *Illuminating sets of constant width*, Mathematika **35** (1988), 280–289.